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The Trouton–Noble paradox

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Abstract. Mechanical and electromagnetic relations involved in the Trouton–Noble paradox are analysed on the basis of special relativity theory as well as on the basis of Maxwellian electrodynamics. It is shown that the paradox only arises when not all dynamical effects associated with the moving capacitor used by Trouton and Noble are considered. Crucial for the resolution of the paradox is the fact that the forces and torques in an electromagnetic system involving moving charge distributions are associated with both the mechanical momentum and electromagnetic momentum of the system. Once the electromagnetic momentum of the moving capacitor is properly taken into account, the paradox disappears.

1. Introduction

In 1903, Trouton and Noble [1] carried out an electromagnetic experiment, originally suggested by Fitzgerald [2], designed to detect the motion of the earth through the ether. Calculations based on considerations of the electromagnetic energy in the moving capacitor appeared to show that, because of the earth's motion relative to the ether, a freely suspended charged parallel-plate capacitor should tend to orient itself so that its plates would be perpendicular to the direction of the earth's velocity. In spite of the great sensitivity of their apparatus, Trouton and Noble observed no tendency of the capacitor to rotate or to assume any preferred orientation.

From the standpoint of Maxwellian electromagnetic theory this result appeared to be inexplicable, and the Trouton and Noble experiment, just as the Michelson–Morley experiment, was partly responsible for the general acceptance of Einstein's special theory of relativity soon after its publication in 1905. According to the relativity principle enunciated by Einstein, since a charged capacitor at rest does not experience a torque, a charged capacitor moving with uniform speed along a straight line should not experience a torque either.

However, there still remained two serious problems with the negative outcome of the Trouton and Noble experiment and with the relativistic explanation of the outcome. First, since Maxwell's equations satisfy the principle of relativity, Maxwellian electrodynamics should lead to the same result as the theory of relativity, and the theory of relativity should not contradict Maxwellian electrodynamics. Second, relativistic transformation equations for forces and torques appeared to indicate that, contrary to the relativity principle, a moving capacitor should experience a torque. Thus the relativity principle provided at best only a partial explanation of the negative result of the Trouton and Noble experiment, and, in the absence of a complete explanation, this result became known as the 'Trouton and Noble paradox'.

Although the Trouton and Noble paradox has been frequently discussed in the literature, and several explanations of the paradox had been proposed [3–5], a complete and definitive

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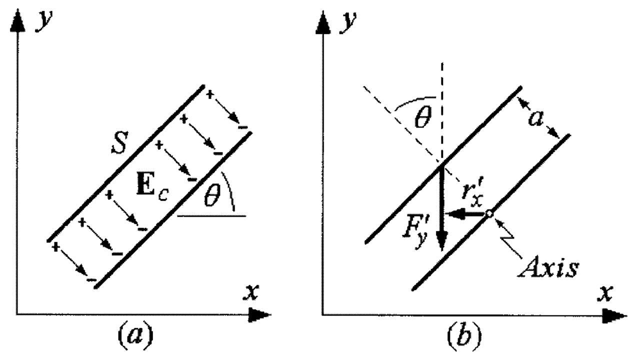

Figure 1. (*a*) A parallel-plate capacitor at rest in a rectangular system of coordinates. (*b*) The force F'_y and the lever arm r'_x in the stationary capacitor are needed for relativistic calculations of the torque acting on a moving capacitor. (The direction of the torque on the moving capacitor is *opposite* to the direction indicated by F'_y and r'_x in this figure; see text.)

resolution of the paradox appears to be either nonexistent or not generally known. In standard books on electromagnetic theory and relativity, the paradox is explained in terms of plausibility arguments based on considering electric and magnetic forces acting between *two point charges* located at the ends of a moving rod; the mechanical or electromagnetic relations in a moving capacitor are not discussed at all [6–11]. And even if the electromagnetic relations in a moving capacitor are discussed, a resolution of the paradox is not provided; instead, the question of the correct expression for the electromagnetic energy in the moving capacitor is debated [12].

Clearly then, a rigorous theoretical analysis of mechanical and electromagnetic relations involved in the Trouton–Noble's experiment is highly desirable. Such an analysis is presented in this paper.

2. The Trouton–Noble paradox as a relativistic (mechanical) paradox

Consider a thin parallel-plate capacitor at rest in an orthogonal system of coordinates as shown in figure 1. According to Trouton and Noble, this capacitor should experience a torque when the capacitor is allowed to move. Let us assume that the capacitor moves with uniform velocity $\boldsymbol{v}$ in the positive direction of the x-axis, and let us find the torque acting on the moving capacitor by using relativistic transformation equations. For this purpose we shall take as the axis of rotation a straight line parallel to the z-axis and passing through the middle of the negative plate. By the symmetry of the system, the torque on the capacitor must then be due to the forces acting on charges located on the positive plate, and the point of application of the resultant of these forces must be the midpoint of this plate. The relativistic transformation equations for the torque are [13]

$$T_x = T'_x/\gamma \tag{1}$$

$$T_y = T'_y + (v^2/c^2) r'_x F'_z \tag{2}$$

$$T_z = T'_z - (v^2/c^2) r'_x F'_y \tag{3}$$

where T_x, T_y, T_z are the components of the torque acting on the moving capacitor, T'_x, T'_y, T'_z are the components of the torque acting on the stationary capacitor, $\gamma = (1 - v^2/c^2)^{-1/2}$, c is the velocity of light, F'_x, F'_y, F'_z are the components of the force acting on the positive plate of the stationary capacitor, and r'_x, r'_y, r'_z are the components of the lever arm joining the axis of rotation with this force. Since the torque on the stationary capacitor is zero, since the force acting on the positive plate of the stationary capacitor (electrostatic attraction to the negative plate) has no z component, and since the force acting on the positive plate of the stationary capacitor does have a y component (see figure 1(*b*)), equations (1)–(3) reduce to a single equation

$$T_z = -(v^2/c^2) r'_x F'_y. \tag{4}$$

We have thus obtained a paradoxical result: contrary to the relativity principle, although our stationary capacitor experiences no torque, the same capacitor moving with uniform velocity along a straight line appears to experience a torque. What makes this result especially surprising is that we have arrived at it by using relativistic transformations that are based on the very same relativity principle with which they now appear to conflict.

To complete our calculations, let us express the torque given by equation (4) in terms of the electric and geometrical quantities pertaining to our capacitor. Let the surface charge density on the capacitor's plates be σ, let it be uniform, let the distance between the capacitor's plates be a, and let the surface area of the capacitor's plates be S. Neglecting edge effects, the electric field produced by the negative plate of the capacitor at the location of the positive plate is

$$\boldsymbol{E}_- = \frac{\sigma}{2\varepsilon_0}(\sin\theta \boldsymbol{i} - \cos\theta \boldsymbol{j}) \tag{5}$$

where ε_0 is the permittivity of space and $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the direction of the x- and y-axis, respectively. The force acting on the positive plate is then

$$\boldsymbol{F} = q\boldsymbol{E}_- = \frac{q\sigma}{2\varepsilon_0}(\sin\theta \boldsymbol{i} - \cos\theta \boldsymbol{j}) = \frac{\sigma^2 S}{2\varepsilon_0}(\sin\theta \boldsymbol{i} - \cos\theta \boldsymbol{j}) \tag{6}$$

where q is the total charge residing on the positive plate. For reasons which will become clear later, we shall rewrite equation (6) as

$$\boldsymbol{F} = \frac{\varepsilon_0 E_c^2 S}{2}(\sin\theta \boldsymbol{i} - \cos\theta \boldsymbol{j}) \tag{7}$$

where E_c is the electric field between the capacitor's plates (observe that $E_c = \sigma/\varepsilon_0$).

For F'_y in equation (4) we then have

$$F'_y = -\frac{\varepsilon_0 E_c^2 S}{2}\cos\theta \tag{8}$$

and for r'_x we have, according to figure 1,

$$r'_x = -a\sin\theta. \tag{9}$$

Substituting equations (8) and (9) into equation (4), we obtain for the torque acting on the moving capacitor

$$\boldsymbol{T} = -\frac{\varepsilon_0 v^2 E_c^2 V}{4c^2}\sin 2\theta \boldsymbol{k} \tag{10}$$

where V is the volume of the capacitor, $\boldsymbol{k}$ is a unit vector along the z-axis, and where we have used the relation $\sin\theta\cos\theta = (\sin 2\theta)/2$. As can be seen from this equation, when the capacitor shown in figure 1 is moving, it should experience a torque causing it to turn *clockwise* and forcing its plates to orient themselves parallel to its line of motion (parallel to the velocity vector $\boldsymbol{v}$).

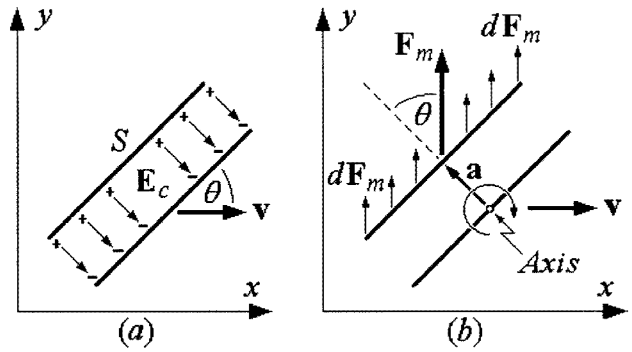

Figure 2. (*a*) A parallel-plate capacitor of surface area S and plate separation a moves with velocity v parallel to the x-axis. (*b*) The negative plate of the moving capacitor creates a magnetic field at the location of the positive plate. The magnetic field exerts a force F_m on the charges carried by the positive plate. This force produces a *clockwise* torque on the capacitor.

3. The Trouton–Noble paradox as an electrodynamic paradox

Trouton and Noble performed their experiment before the advent of the special relativity theory, and their calculations were based on the prerelativistic Maxwellian electrodynamics. Let us see what Maxwellian electrodynamics can tell us about the torque acting on a moving capacitor. Consider a thin parallel-plate capacitor (figure 2(*a*)) moving with velocity $\boldsymbol{v} = v\boldsymbol{i}$ relative to an orthogonal system of coordinates. Let the surface charge density on the capacitor's plates be σ, let it be uniform, and let the surface area of the capacitor's plates be S. Neglecting edge effects, the electric field produced by the negative plate of the capacitor at the location of the positive plate is

$$\boldsymbol{E}_- = \frac{\sigma}{2\varepsilon_0}(\sin\theta\boldsymbol{i} - \cos\theta\boldsymbol{j}). \tag{11}$$

Since $\boldsymbol{E}_-$ is moving, it creates a magnetic flux density field [14]

$$\boldsymbol{B}_- = \frac{\boldsymbol{v}\times\boldsymbol{E}_-}{c^2} \tag{12}$$

or, using equation (11),

$$\boldsymbol{B}_- = -\boldsymbol{k}\frac{v\sigma}{2\varepsilon_0 c^2}\cos\theta. \tag{13}$$

Moving through this field, a charge element $\sigma\,\mathrm{d}S$ on the positive plate of the capacitor experiences a magnetic force (Lorentz force)

$$\mathrm{d}\boldsymbol{F}_m = \sigma\,\mathrm{d}S(\boldsymbol{v}\times\boldsymbol{B}_-) = \boldsymbol{j}\frac{v^2\sigma^2}{2\varepsilon_0 c^2}\cos\theta\,\mathrm{d}S \tag{14}$$

or

$$\mathrm{d}\boldsymbol{F}_m = \boldsymbol{j}\frac{\varepsilon_0 v^2 E_c^2}{2c^2}\cos\theta\,\mathrm{d}S \tag{15}$$

where $E_c = \sigma/\varepsilon_0$ is the electric field between the capacitor's plates. As we shall presently see, the force given by equation (15) creates a torque on the moving capacitor.

To find the torque, let us take as the instantaneous axis of rotation a line passing through the middle of the negative plate parallel to the z-axis; observe that the electric forces acting on the two plates create no torque relative to this axis because, by the symmetry of the system, the resultant electric force on each plate passes through this axis. Integrating equation (15) over the surface of the positive plate, we obtain for the total magnetic force acting on this plate

$$\boldsymbol{F}_m = \boldsymbol{j} \int \frac{\varepsilon_0 v^2 E_c^2}{2c^2} \cos\theta \, \mathrm{d}S = \frac{\varepsilon_0 v^2 E_c^2 S}{2c^2} \cos\theta \boldsymbol{j}. \tag{16}$$

By the symmetry of the system, this force, being the resultant of the magnetic forces acting on all the surface elements of the positive plate, is applied to the centre of the plate, as shown in figure 2(*b*). The lever arm of this force with respect to the instantaneous axis is

$$\boldsymbol{a} = -a \sin\theta \boldsymbol{i} + a \cos\theta \boldsymbol{j}. \tag{17}$$

By equations (16) and (17), the torque acting on the positive plate is therefore

$$\boldsymbol{T} = \boldsymbol{a} \times \boldsymbol{F}_m = -\frac{a\varepsilon_0 v^2 E_c^2 S}{2c^2} \sin\theta \cos\theta \boldsymbol{k} \tag{18}$$

or

$$\boldsymbol{T} = -\frac{\varepsilon_0 v^2 E_c^2 V}{4c^2} \sin 2\theta \boldsymbol{k} \tag{19}$$

where $V = aS$ is the volume of the capacitor and $\boldsymbol{k}$ is a unit vector in the direction of the z-axis. (There is also a magnetic force acting on the negative plate. It is equal and opposite to the magnetic force acting on the positive plate. However, it creates no torque relative to the instantaneous axis because the resultant of this force is applied to the centre of the negative plate and therefore passes through the axis.) Observe that equation (19) is exactly the same as equation (10) which we found in section 2 by using very simple relativistic transformations†.

Thus, also according to direct nonrelativistic electromagnetic calculations, the moving capacitor should experience a torque forcing its plates to orient themselves parallel to $\boldsymbol{v}$.

4. The resolution of the paradox

An interaction of moving charges is a complex process involving not only the electric and magnetic fields at the location of the charges but also the entire electromagnetic field of the system. This electromagnetic field is a carrier of electromagnetic momentum. As the capacitor moves, the electromagnetic field created by the capacitor's charges changes its position in space, and so does the electromagnetic momentum. The rate of change of electromagnetic momentum at a point is equivalent to an additional force acting at this point. Thus, in order to obtain a complete picture of the dynamical effects associated with the moving capacitor, we must take into account not only the explicit electric and magnetic forces acting on the charges carried by the capacitor, but also the system's electromagnetic momentum. In so doing, however, we must differentiate between the field-producing and field-experiencing charges as well as between the electric and magnetic fields associated with the field-producing and the field-experiencing charges [15].

The density of electromagnetic momentum in an electromagnetic field is [16]

$$\boldsymbol{g} = \frac{\boldsymbol{E} \times \boldsymbol{H}}{c^2} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0 c^2} = \varepsilon_0 \boldsymbol{E} \times \boldsymbol{B} \tag{20}$$

where μ_0 is the permeability of space and where we have used $\mu_0 \varepsilon_0 = 1/c^2$. We are interested in the electromagnetic momentum created by the interaction of the magnetic field of the negative plate with the electric field of the positive plate of the capacitor, since the torque that we have calculated in sections 2 and 3 is due to the forces acting on the positive plate.

† This serves as an excellent example of the power of relativistic transformations and of the compatibility of Maxwellian electrodynamics with special relativity theory.

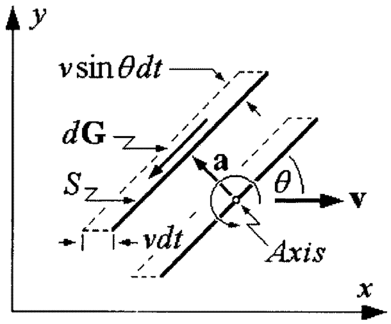

Figure 3. As the capacitor moves, the electromagnetic momentum $\boldsymbol{G}$ at the positive plate of the capacitor changes and produces a *counterclockwise* torque on the capacitor. Since this torque counterbalances the torque produced by the magnetic force acting on the plate, the capacitor does not turn.

The electric field produced by the positive plate in front of itself is

$$\boldsymbol{E}_{+f} = \frac{\sigma}{2\varepsilon_0}(\sin\theta \boldsymbol{i} - \cos\theta \boldsymbol{j}) \tag{21}$$

and the electric field produced by the positive plate behind itself is

$$\boldsymbol{E}_{+b} = -\frac{\sigma}{2\varepsilon_0}(\sin\theta \boldsymbol{i} - \cos\theta \boldsymbol{j}). \tag{22}$$

Using equations (20), (21), and (13), we obtain for the density of the electromagnetic momentum associated with $\boldsymbol{E}_{+f}$ and $\boldsymbol{B}_-$ in front of the positive plate

$$\boldsymbol{g}_f = \varepsilon_0(\boldsymbol{E}_{+f} \times \boldsymbol{B}_-) = \frac{v\sigma^2(\cos^2\theta \boldsymbol{i} + \cos\theta \sin\theta \boldsymbol{j})}{4\varepsilon_0 c^2} \tag{23}$$

which we can write as

$$\boldsymbol{g}_f = \frac{\varepsilon_0 v E_c^2 \cos\theta(\cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j})}{4c^2}. \tag{24}$$

Observe that $\boldsymbol{g}_f$ is directed upward along the positive plate and is perpendicular to $\boldsymbol{a}$.

Using equations (20), (22) and (13), we similarly obtain for the density of the electromagnetic momentum associated with $\boldsymbol{E}_{+b}$ and $\boldsymbol{B}_-$ in the back of the positive plate

$$\boldsymbol{g}_b = -\frac{\varepsilon_0 v E_c^2 \cos\theta(\cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j})}{4c^2}. \tag{25}$$

Observe that $\boldsymbol{g}_b$ is directed downward along the positive plate and is perpendicular to $\boldsymbol{a}$.

As the capacitor moves, a region of space originally in front of the positive plate is now behind this plate, and the density of electromagnetic momentum in this region of space changes from that given by equation (24) to that given by (25). The total change of the momentum density in this region (final density minus initial density) is therefore

$$\boldsymbol{g} = \boldsymbol{g}_{+b} - \boldsymbol{g}_{+f} = -\frac{\varepsilon_0 v E_c^2 \cos\theta(\cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j})}{2c^2}. \tag{26}$$

Note that $\boldsymbol{g}$ is directed downward along the positive plate and is perpendicular to $\boldsymbol{a}$.

During a time interval $\mathrm{d}t$ the capacitor moves through a distance $\mathrm{d}x = v\,\mathrm{d}t$. The volume of space adjacent to the positive plate in which the density of the electromagnetic momentum changes during this time interval is therefore (see figure 3)

$$\mathrm{d}V = vS\sin\theta\,\mathrm{d}t \tag{27}$$

and the electromagnetic momentum that has experienced a change during this time interval is, by equations (26) and (27),

$$\mathrm{d}\boldsymbol{G} = \boldsymbol{g}\,\mathrm{d}V = -\frac{\varepsilon_0 v^2 E_c^2 S\sin\theta\cos\theta(\cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j})}{2c^2}\,\mathrm{d}t. \tag{28}$$

According to equation (28), the magnitude of d$\boldsymbol{G}$ is simply (note that $\cos^2\theta + \sin^2\theta = 1$)

$$\mathrm{d}G = \frac{\varepsilon_0 v^2 E_c^2 S \sin\theta \cos\theta}{2c^2}\,\mathrm{d}t = \frac{\varepsilon_0 v^2 E_c^2 S \sin 2\theta}{4c^2}\,\mathrm{d}t. \tag{29}$$

Since $\boldsymbol{g}$, and therefore d$\boldsymbol{G}$, are perpendicular to $\boldsymbol{a}$, the *angular* electromagnetic momentum that has experienced a change during the time interval dt evaluated with respect to the instantaneous axis defined above is

$$\mathrm{d}\boldsymbol{I} = \boldsymbol{a} \times \mathrm{d}\boldsymbol{G} = a\,\mathrm{d}G\boldsymbol{k} = \frac{a\varepsilon_0 v^2 E_c^2 S \sin 2\theta}{4c^2}\,\mathrm{d}t\,\boldsymbol{k} = \frac{\varepsilon_0 v^2 E_c^2 V \sin 2\theta}{4c^2}\,\mathrm{d}t\,\boldsymbol{k} \tag{30}$$

where $V = aS$ is the volume of the capacitor. The rate of change of the angular electromagnetic momentum is therefore

$$\frac{\partial \boldsymbol{I}}{\partial t} = \frac{\varepsilon_0 v^2 E_c^2 V}{4c^2}\sin 2\theta\boldsymbol{k}. \tag{31}$$

(Of course, the electromagnetic momentum also changes in the space adjacent to the negative plate of the moving capacitor. However, since the momentum in this space, just as the momentum in the space adjacent to the positive plate, is parallel to the plate, and since the volume element in which the momentum experiences a change is adjacent to the negative plate and thus is adjacent to the axis of rotation, there is no angular momentum associated with the change of electromagnetic momentum there.)

The torque given by equations (10) and (19), and hence the rate of change of the *angular mechanical momentum* associated with this torque, is equal in magnitude and opposite in direction to the rate of change of the *angular electromagnetic field momentum* given by equation (31). Clearly then, the rate of change of the angular electromagnetic field momentum associated with the moving capacitor completely balances the rate of change of the angular mechanical momentum of the capacitor, conserves the *total angular momentum of the systems*, and prevents the capacitor from rotating†. Needless to say that neither the torque nor the rate of change of the angular momentum depends on the location of the axis of rotation, so that our result is perfectly general.

5. Discussion

The calculations presented above, based on relativistic transformations as well as on Maxwellian electromagnetic theory, show that the capacitor in the Trouton–Noble experiment should not rotate. Thus there is no disagreement between the results of the Trouton–Noble experiment and the theory of the experiment.

However, there is a disagreement between our calculations and those of Trouton and Noble. First, the direction of the torque given by our equations (10) and (19) is opposite to that found by Trouton and Noble. Second, the magnitude of the torque given by our equations (10) and (19) is only $\frac{1}{2}$ of the magnitude found by Trouton and Noble. There is, of course, little doubt that Trouton and Noble's calculations, which were based on the presumed variation of the electromagnetic energy in the moving capacitor, are wrong. The precise error of their calculations is, however, somewhat uncertain. Generally speaking, it is based on the incorrect assumption that the energy of the moving capacitor is increased due to the appearance of the magnetic energy associated with the magnetic field in the capacitor. Actually, the magnetic

† An example of an opposite relation between the mechanical and electromagnetic momentum is presented by the 'Feynman's disc paradox.' There, a charged disc, initially at rest, begins to rotate and thus acquires a mechanical angular momentum at the expense of the initially present but disappearing angular electromagnetic field momentum. See [17].

field *decreases* the electromagnetic energy in the capacitor by weakening the attraction between the capacitor's plates. Furthermore, the exact expression for the electromagnetic energy in the moving capacitor appears to be questionable, and different authors have suggested different expressions for it‡.

Until the energy question is resolved, we cannot give a definitive analysis of the Trouton and Noble calculations. However, as far as the Trouton–Noble paradox is concerned, such an analysis is not needed. The paradox does not really exist. As has been shown in this paper, there is no disagreement between the principle of relativity, relativistic transformations, Maxwellian electromagnetic theory, and the negative outcome of the Trouton–Noble experiment.

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‡ See [12].